L^2-Contraction of Planar Traveling Waves in the 2D Dissipative KP Equation

Harish Ravichandar

Department of Mathematics
University of Kansas

Joint work with Geng Chen and Yannan Shen

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Photographs: Coastal Wiki.

What happens to a perturbed planar shock?

Dissipative Kadomtsev–Petviashvili equation on $\mathbb{R} \times \mathbb{T}^1$

$$(u_t + uu_x - \epsilon_1 u_{xx} - \epsilon_2 u_{yy} + \delta u_{xxx})_x + \lambda u_{yy} = 0, \quad \lambda = \pm 1.$$

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Perturbation: changes in wave height and transverse front shape, with the same far-field states.
Question: does L^2-contraction hold for large, y-dependent perturbations after a shift?

The planar shock and its propagation speed

Here a shock is a smooth traveling transition between two constant states $u_- > u_+$. Seek a planar solution with an initially unknown speed:

$\tilde{u}(x, y, t) = u(x - \sigma t), \qquad \xi = x - \sigma t, \qquad u(\pm\infty) = u_\pm.$

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$$-\sigma(u - u_+) + \frac{1}{2}(u^2 - u_+^2) = \epsilon_1 u' - \delta u''.$$

The left end state determines the speed

$$-\sigma(u_- - u_+) + \frac{1}{2}(u_-^2 - u_+^2) = 0 \qquad \Longrightarrow \qquad \boxed{\sigma = \frac{u_- + u_+}{2}}.$$

Here $u(\xi)$ is the profile. In the remaining slides, the profile is denoted by $\tilde{u}(\xi)$, as in the manuscript.

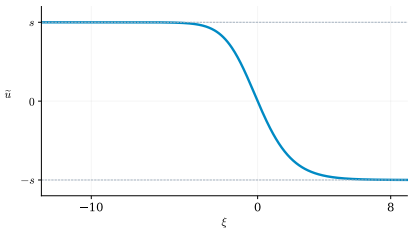
Monotone and oscillatory shock profiles

$$u_- = -u_+ = s, \quad \sigma = 0, \qquad \epsilon_1 \widetilde{u}' - \delta \widetilde{u}'' = \frac{1}{2}(\widetilde{u}^2 - s^2), \quad \widetilde{u}(\pm\infty) = \mp s.$$

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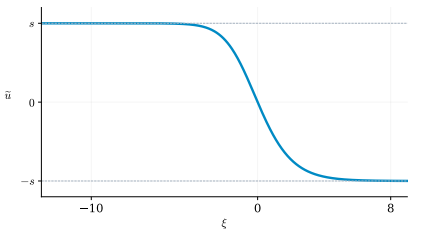


Nonlinear steepening and smoothing sustain the front.
Stability asks whether perturbations remain controlled.

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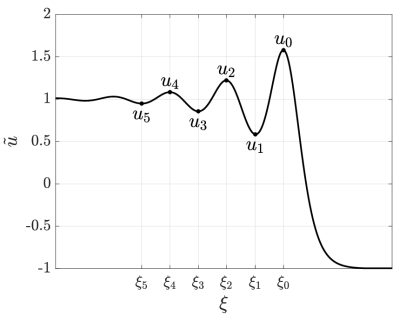
$$u_- = -u_+ = s, \quad \sigma = 0, \qquad \epsilon_1 \widetilde{u}' - \delta \widetilde{u}'' = \frac{1}{2}(\widetilde{u}^2 - s^2), \quad \widetilde{u}(\pm\infty) = \mp s.$$

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Nonlinear steepening and smoothing sustain the front.
Stability asks whether perturbations remain controlled.

Oscillatory: $\delta s/\epsilon_1^2 > 1/4$



Dispersion produces the oscillatory tail.

Left: profile ODE, $\epsilon_1 = s = 1$, $\delta = 0.2$. Right: Chen–Eun–Kang–Shen [8], Fig. 1; $\delta = 10$, $\epsilon_1 = s = 1$.
The published oscillatory example illustrates the geometry and lies outside the smaller contraction-theorem range.

Global Existence and Uniqueness

Let $f \in C^\infty(\mathbb{R})$ be monotone, with $f = u_-$ for $x \le -1$ and $f = u_+$ for $x \ge 1$. On $\Omega = \mathbb{R} \times \mathbb{T}^1$, assume

$$u_0 - f \in H^s(\Omega), \quad s > 1, \qquad \int_{\mathbb{R}} (u_0 - f)\, dx = 0 \quad (y \in \mathbb{T}^1).$$

For every $T > 0$, there is a unique KP solution $u \in \bar{X}_T$ with $u(0) = u_0$, where

$$\bar{X}_T := \left\{ u : \begin{array}{c} u - f \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ \displaystyle\int_{\mathbb{R}} \big(u(t,x,y) - f(x)\big)\, dx = 0, \quad t \in [0,T],\ y \in \mathbb{T}^1 \end{array} \right\}.$$

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Proof idea

1. **Standard semigroup theory** gives the linear evolution and Duhamel formula.
2. The **contraction fixed-point theorem** gives local existence and uniqueness.
3. A priori H^1 control and a **continuation argument** give global existence.

Monotone shocks: L^2-contraction

Let u be the global KP solution, $u_- - u_+ = 2s > 0$, and $\epsilon_1, \epsilon_2 > 0$. Assume

$$0 < \frac{\delta s}{\epsilon_1^2} \le \frac{1}{4}, \qquad M = \frac{1}{2}, \qquad \epsilon_1\epsilon_2 > \frac{s^2}{16\pi^2}.$$

Define the Lipschitz shift X (displacement of the front from σt) by

$$\dot{X}(t) = -\frac{M}{2s}\int_\Omega \big(u(t, x + X(t), y) - \widetilde{u}(x - \sigma t)\big)\widetilde{u}'(x - \sigma t)\, dx\, dy, \qquad X(0) = 0.$$

Then, for every $t \ge 0$,

$$\int_\Omega |u(t, x, y) - \widetilde{u}(x - \sigma t - X(t))|^2\, dx\, dy \le \int_\Omega |u_0 - \widetilde{u}|^2\, dx\, dy.$$

The shift velocity vanishes asymptotically

$$|\dot{X}(t)| \longrightarrow 0 \qquad \text{as } t \to \infty.$$

MAIN THEOREM 2 [KP, Theorem 1.4, (1.14); Remark 1.5]

Oscillatory shocks: L^2-contraction

Let u be the global KP solution, $u_- = -u_+ = s > 0$, and $\epsilon_1, \epsilon_2 > 0$. Assume

$$\frac{1}{4} < \frac{\delta s}{\epsilon_1^2} < \frac{1}{2}, \qquad M = \frac{4}{3}, \qquad s \le \pi \sqrt{\frac{7.86\epsilon_1\epsilon_2}{M}}.$$

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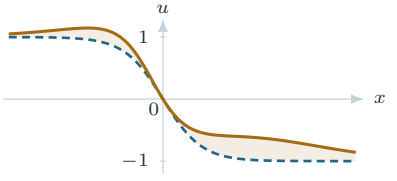
Burgers: why the shift is needed

$$u_t + uu_x = u_{xx}, \qquad \widetilde{u}(x) = -\tanh(x/2), \qquad u_- = 1, \quad u_+ = -1.$$

A smooth mass perturbation moves the Burgers front from $x = 0$ toward $x = 2$.

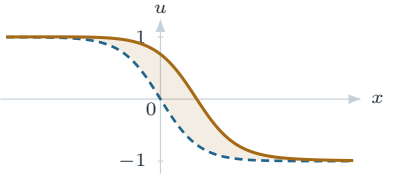
Amber: exact Burgers solution. Dashed blue: reference. All panels use the same axes.

At $t = 0$: initial perturbation



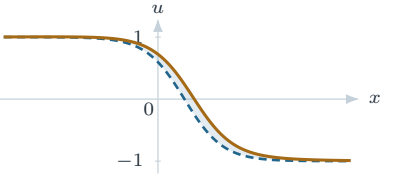
$\|u_0 - \widetilde{u}\|_2^2 \simeq 0.91$

At $t = 12$: fixed reference



$\|u(12) - \widetilde{u}\|_2^2 \simeq 2.25$

At $t = 12$: shift $X(12) \simeq 1.37$



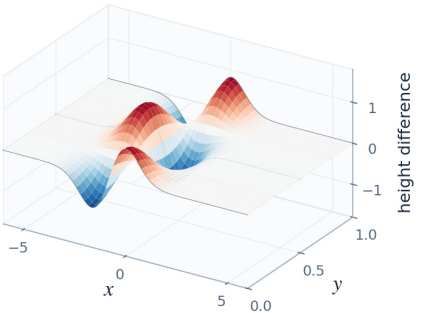
$\|u(12) - \widetilde{u}(\cdot - X(12))\|_2^2 \simeq 0.19$

$$\dot{X}(t) = -\frac{1}{4}\int_{\mathbb{R}} \big(u(t, x + X(t)) - \widetilde{u}(x)\big)\widetilde{u}'(x)\,dx, \qquad X(0) = 0.$$

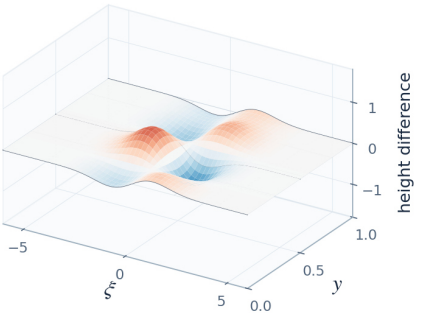
Control of the two-dimensional perturbation

$$w_0 = u_0 - \widetilde{u}, \qquad w(t,\xi,y) = u(t,\xi+\sigma t + X(t),y) - \widetilde{u}(\xi).$$

Initial perturbation w_0



Later perturbation $w(t)$, aligned



$$\|w(t)\|^2_{L^2(\Omega)} \le \|w_0\|^2_{L^2(\Omega)}, \qquad t \ge 0.$$

Same axes and color scale. Illustrative profiles, not computed KP time evolution; no particular shape or decay rate is asserted.

The equation in the moving frame

Set $\xi = x - \sigma t$ and retain the letter u for the solution in moving coordinates. Then the KP system becomes

$$\begin{cases} u_t - \sigma u_\xi + u u_\xi = \epsilon_1 u_{\xi\xi} + \epsilon_2 u_{yy} - \delta u_{\xi\xi\xi} + v, \\ v_\xi = -\lambda u_{yy}. \end{cases}$$

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For a Lipschitz shift $X(t)$, define

$$u^X(t,\xi,y) = u(t,\xi + X(t), y), \qquad w = u^X - \widetilde{u},$$
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Subtracting the profile equation gives

$$w_t - \sigma w_\xi + (\widetilde{u} w)_\xi + w w_\xi = \epsilon_1 w_{\xi\xi} + \epsilon_2 w_{yy} - \delta w_{\xi\xi\xi} + v^X + \dot{X}(t)(w_\xi + \widetilde{u}').$$

This yields the relative-energy identity on the next slide.

Relative energy and the choice of shift

For an arbitrary Lipschitz shift X, the relative-energy identity gives

$$\frac{d}{dt}\int_\Omega \frac{|u-\widetilde{u}^{-X}|^2}{2} = \dot{X}(t)\int_\Omega w\widetilde{u}' - \frac{1}{2}\int_\Omega w^2\widetilde{u}' - \epsilon_1\int_\Omega |w_\xi|^2 - \epsilon_2\int_\Omega |w_y|^2.$$

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Define the shift by

$$\dot{X}(t) = -\frac{M}{2s}\int_\Omega w\widetilde{u}'\,d\xi\,dy, \quad X(0)=0, \qquad M = \frac{1}{2} \text{ or } \frac{4}{3}.$$

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Substitution gives

$$\frac{d}{dt}\int_\Omega \frac{|u-\widetilde{u}^{-X}|^2}{2} = -\frac{M}{2s}\left(\int_\Omega w\widetilde{u}'\right)^2 - \frac{1}{2}\int_\Omega w^2\widetilde{u}' - \epsilon_1\int_\Omega |w_\xi|^2 - \epsilon_2\int_\Omega |w_y|^2.$$

Here the dispersive terms and the KP term $\int_\Omega wv^X$ vanish under the stated compatibility and periodicity conditions.

The global profile coordinate and energy in W

Center the states at $u_- = -u_+ = s$. Since $\widetilde{u}' < 0$, the profile defines a global transformation $\mathbb{R} \to (0, 2s)$:

$$z = s - \widetilde{u}(\xi), \quad \frac{dz}{d\xi} = -\widetilde{u}' > 0, \quad W(z, y) = w(\xi, y), \quad \overline{W}(y) = \int_0^{2s} W(z, y)\, dz.$$

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The relative-energy inequality in W, for $M = 1/2$, is

$$\frac{d}{dt} \int_\Omega \frac{|u - \widetilde{u}^{-X}|^2}{2} \leq \frac{1}{2} \int_{\mathbb{T}^1} \int_0^{2s} W^2\, dz\, dy - \lambda_1 \int_{\mathbb{T}^1} \int_0^{2s} z(2s - z)|W_z|^2\, dz\, dy$$
$$- \frac{1}{4s} \left(\int_{\mathbb{T}^1} \overline{W}\, dy \right)^2 - \epsilon_2 \int_{\mathbb{T}^1} \int_0^{2s} \frac{|W_y|^2}{dz/d\xi}\, dz\, dy.$$

The longitudinal diffusion term uses the profile slope bound with $\lambda_1 > 1/4$ [KP, (4.28), (4.41); 7, Lemma 2.3]. The reciprocal Jacobian appears in transverse diffusion.

The remaining positive W^2 term is the one to which we apply weighted Poincaré.

Kang–Vasseur Poincaré and its application

Kang–Vasseur weighted Poincaré: general interval form

For $a < b$ and sufficiently regular $w : [a, b] \to \mathbb{R}$ with $\int_a^b (z-a)(b-z)|w_z|^2\,dz < \infty$,

$$\int_a^b w^2\,dz \le \frac{1}{2}\int_a^b (z-a)(b-z)|w_z|^2\,dz + \frac{1}{b-a}\left(\int_a^b w\,dz\right)^2.$$

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Here $a = 0$, $b = 2s$, and $w(z) = W(z, y)$ for each fixed y

Apply the inequality, divide by 2, and integrate in y:

$$\frac{1}{2}\int_{\mathbb{T}^1}\int_0^{2s} W^2\,dz\,dy \le \frac{1}{4}\int_{\mathbb{T}^1}\int_0^{2s} z(2s-z)|W_z|^2\,dz\,dy + \frac{1}{4s}\int_{\mathbb{T}^1} \overline{W}^2\,dy.$$

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$$\frac{1}{2}\int_{\mathbb{T}^1}\int_0^{2s} W^2\,dz\,dy \le \frac{1}{4}\int_{\mathbb{T}^1}\int_0^{2s} z(2s-z)|W_z|^2\,dz\,dy + \frac{1}{4s}\int_{\mathbb{T}^1}\overline{W}^2\,dy.$$

Combine with the energy inequality, using $\lambda_1 > 1/4$

$$\frac{d}{dt}\int_\Omega \frac{|u - \widetilde{u}^{-X}|^2}{2} \le \frac{1}{4s}\int_{\mathbb{T}^1}\left|\overline{W} - \int_{\mathbb{T}^1}\overline{W}\,dy\right|^2 dy - \epsilon_2\int_{\mathbb{T}^1}\int_0^{2s}\frac{|W_y|^2}{dz/d\xi}\,dz\,dy.$$

The longitudinal weighted gradient has a nonpositive coefficient. Only the transverse variance remains.

Periodic Poincaré controls the slice mean

The remaining term is the transverse variation of $\overline{W}$. In (ξ, y) coordinates, transverse diffusion is $\epsilon_2 \int_\Omega |w_y|^2$.

Periodic Poincaré on $\mathbb{T}^1$ (period 1), used in [KP, (4.43)]

$$\int_{\mathbb{T}^1} \left| \overline{W} - \int_{\mathbb{T}^1} \overline{W}\, dy \right|^2 dy \le \frac{1}{4\pi^2} \int_{\mathbb{T}^1} |\overline{W}_y|^2\, dy.$$

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Cauchy–Schwarz and the profile bound [KP, Lemma 4.2, (4.35)]

$$\int_{\mathbb{T}^1} |\overline{W}_y|^2\, dy \le \left(\int_{\mathbb{R}} |\widetilde{u}'|^2\, d\xi \right) \int_\Omega |w_y|^2 \le \frac{s^3}{\epsilon_1} \int_\Omega |w_y|^2.$$

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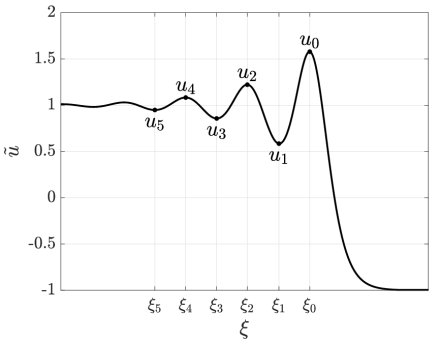
$$\int_{\mathbb{T}^1} |\overline{W}_y|^2\, dy \le \left(\int_{\mathbb{R}} |\widetilde{u}'|^2\, d\xi \right) \int_\Omega |w_y|^2 \le \frac{s^3}{\epsilon_1} \int_\Omega |w_y|^2.$$

Combining these estimates with the energy inequality gives

$$\begin{aligned} \frac{d}{dt} \int_\Omega \frac{|u - \widetilde{u}^{-X}|^2}{2} &\le \frac{1}{4s} \int_{\mathbb{T}^1} \left| \overline{W} - \int_{\mathbb{T}^1} \overline{W}\, dy \right|^2 dy - \epsilon_2 \int_\Omega |w_y|^2 \\ &\le \left(\frac{s^2}{16\pi^2 \epsilon_1} - \epsilon_2 \right) \int_\Omega |w_y|^2 \le 0. \end{aligned}$$

Use $\epsilon_1 \epsilon_2 > s^2/(16\pi^2)$ and integrate in time.

Oscillations require branchwise coordinates



The derivative $\widetilde{u}'$ changes sign and vanishes at the extrema. The same profile value can occur on several branches.

Use the profile as a coordinate on each monotone interval $(\xi_i, \xi_{i-1}), \quad \widetilde{u}(\xi_i) = u_i$.

The profile term has two signs:

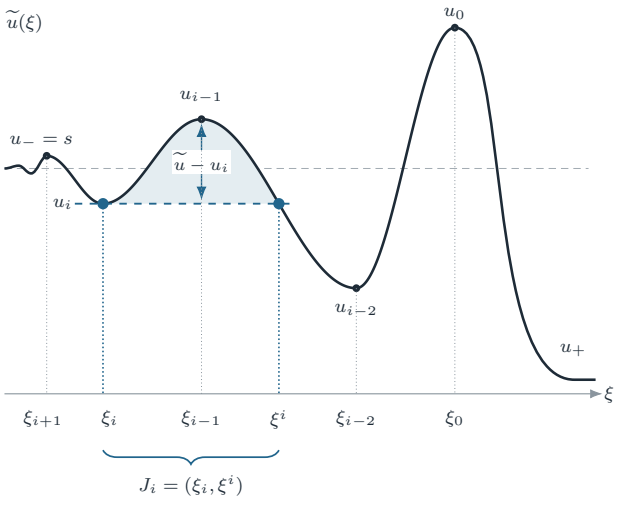
$-\frac{1}{2}w^2\widetilde{u}' \le 0 \quad (\widetilde{u}' > 0),$

$-\frac{1}{2}w^2\widetilde{u}' \ge 0 \quad (\widetilde{u}' < 0).$

Chen–Eun–Kang–Shen, arXiv:2602.22652, Fig. 1. Published example: $\delta = 10$, $\epsilon_1 = s = 1$ (outside the theorem range).

Following Chen–Eun–Kang–Shen [8] for KdV–Burgers, use equal-height overlaps to transfer favorable control across adverse branches.

Equal-height overlaps control branch means



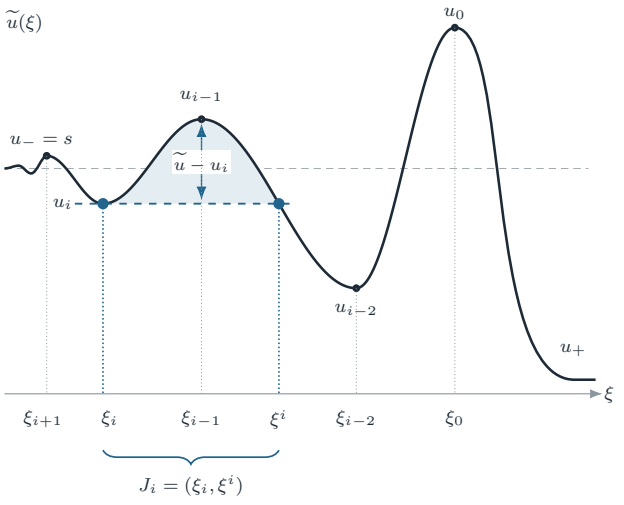
$i = 3$ shown. Rescaled schematic of [KP, Proposition 5.2].

Equal-height overlap

$J_i = (\xi_i, \xi^i), \qquad \xi^i \in (\xi_{i-1}, \xi_{i-2}).$

$\widetilde{u}(\xi_i) = \widetilde{u}(\xi^i) = u_i.$

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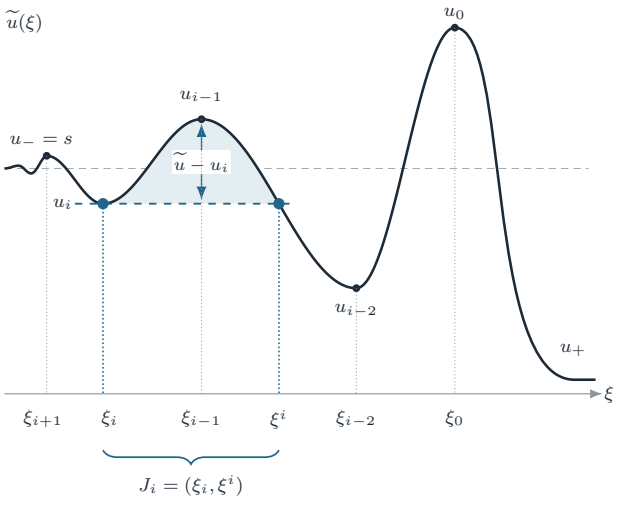
$\widetilde{u}(\xi_i) = \widetilde{u}(\xi^i) = u_i.$

Integration by parts

$$\int_{J_i} w\widetilde{u}'\, d\xi = -\int_{J_i} w_\xi(\widetilde{u} - u_i)\, d\xi.$$

The boundary term vanishes at the two blue points.

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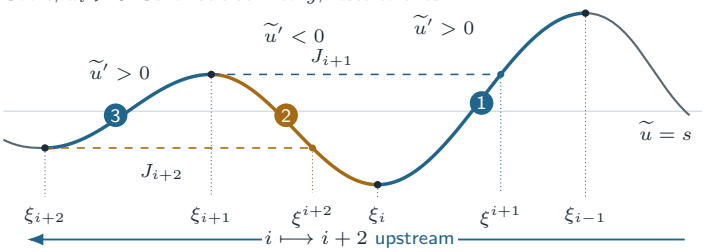
Cauchy–Schwarz

$$\left|\int_{J_i} w\widetilde{u}'\, d\xi\right|^2 \le \left(\int_{J_i} (\widetilde{u} - u_i)^2\, d\xi\right) \times \left(\int_{J_i} |w_\xi|^2\, d\xi\right).$$

The profile factor is controlled by [KP, Proposition 5.2; 8, Proposition 4.2].

The induction step across two branches

Odd i, $a_i > 0$. Schematic at fixed y, rescaled axes.



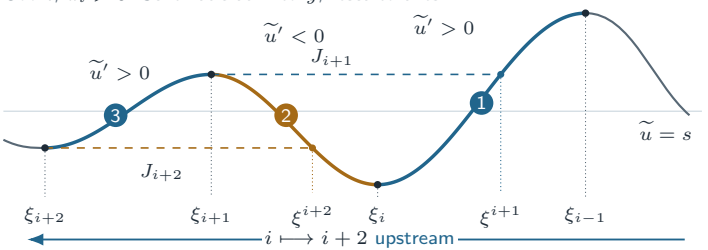
1. Induction hypothesis

$$-\left(\tfrac{1}{2} + a_i\right) \int_{\xi_i}^{\xi_{i-1}} w^2 \widetilde{u}' \, d\xi \le 0.$$

The previous remainder and overlap diffusion control a squared branch mean. Weighted Poincaré then gives the control below.

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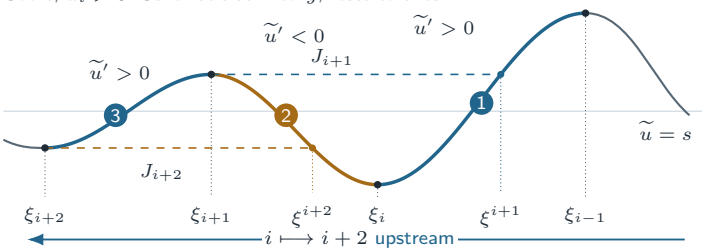
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2. Decreasing branch: use J_{i+1} diffusion and weighted Poincaré

$$\underbrace{\left(\tfrac{1}{2} + a_{i+1}\right) \int_{\xi_{i+1}}^{\xi_i} w^2 \widetilde{u}' \, d\xi}_{\text{transferred control}} \quad \underbrace{-\tfrac{1}{2} \int_{\xi_{i+1}}^{\xi_i} w^2 \widetilde{u}' \, d\xi}_{\text{positive energy term}} = \underbrace{a_{i+1} \int_{\xi_{i+1}}^{\xi_i} w^2 \widetilde{u}' \, d\xi}_{\text{negative remainder}} \le 0.$$

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3. Increasing branch: use J_{i+2} diffusion and weighted Poincaré

$$\underbrace{-a_{i+2} \int_{\xi_{i+2}}^{\xi_{i+1}} w^2 \widetilde{u}' \, d\xi}_{\text{transferred control}} - \tfrac{1}{2} \int_{\xi_{i+2}}^{\xi_{i+1}} w^2 \widetilde{u}' \, d\xi = \underbrace{-\left(\tfrac{1}{2} + a_{i+2}\right) \int_{\xi_{i+2}}^{\xi_{i+1}} w^2 \widetilde{u}' \, d\xi}_{\text{same hypothesis at } i+2} \le 0.$$

Closing the oscillatory contraction estimate

Theorem 5.4: the longitudinal estimate, with $M = 4/3$

$$-\frac{M}{2s}\int_{\mathbb{T}^1}\left(\int_{\mathbb{R}} w\widetilde{u}'\,d\xi\right)^2 dy - \frac{1}{2}\int_{\Omega} w^2\widetilde{u}' - \frac{9}{10}\epsilon_1\int_{\Omega}|w_\xi|^2 \leq 0.$$

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Relative energy leaves a transverse variance. Periodic Poincaré and Cauchy–Schwarz give

$$\int_{\mathbb{T}^1}\left|\int_{\mathbb{R}} w\widetilde{u}'\,d\xi - \int_\Omega w\widetilde{u}'\right|^2 dy \le \frac{1}{4\pi^2}\left(\int_{\mathbb{R}}|\widetilde{u}'|^2\,d\xi\right)\int_\Omega |w_y|^2.$$

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Using the profile identity $\epsilon_1\int_{\mathbb{R}}|\widetilde{u}'|^2\,d\xi = 2s^3/3$, we obtain

$$\frac{d}{dt}\int_\Omega \frac{|u-\widetilde{u}^{-X}|^2}{2} \le \left(\frac{Ms^2}{12\pi^2\epsilon_1} - \epsilon_2\right)\int_\Omega |w_y|^2 \le 0.$$

The amplitude restriction in Main Theorem 2 makes the coefficient nonpositive.

The same energy argument for ZK–Burgers

On $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$, $n \geq 2$, consider

$$u_t + uu_{x_1} - \varepsilon \Delta u + \delta u_{x_1 x_1 x_1} + \kappa \sum_{i=2}^{n} u_{x_1 x_i x_i} = 0.$$

The planar profile is unchanged. For each mixed dispersive term, integration by parts gives

$$\int_\Omega w\, w_{\xi x_i x_i} = -\frac{1}{2} \int_\Omega \partial_\xi |w_{x_i}|^2 = 0, \qquad 2 \leq i \leq n.$$

Thus dispersion contributes no relative energy. Set $\epsilon_1 = \epsilon_2 = \varepsilon$ and use the same longitudinal estimates. Periodic Poincaré on the unit transverse torus controls the remaining slice variance.

The contraction argument applies to regular ZKB solutions on their interval of existence.

Can transverse viscosity be removed?

$$\epsilon_2 u_{yy} \longrightarrow 0$$

The obstacle is explicit: the proof leaves a transverse variance, currently absorbed by $\epsilon_2 \int_\Omega |w_y|^2$. A new coercive mechanism or a transverse phase is needed.

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Mach reflection for KP?

Can contraction be extended to KP Mach-reflection patterns? [3] This would require a nonplanar reference and control of the moving interaction region.

Current result: large perturbations contract around both monotone and oscillatory shocks, up to the moving front location.

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